Even-hole-free graphs of large treewidth

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EHF graphs is a class of graphs not *containing* even hole

Remark. An EHF graph may contains pyramid, but no theta, prism, and even-wheel.

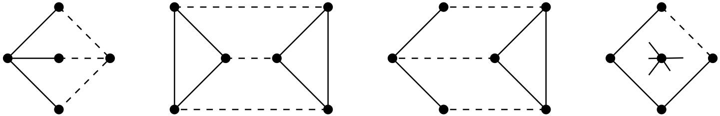

Figure: theta, prism, and pyramid (dashed edge: path of length ≥ 1)

Connectivity of EHF graphs

- Bounded treewidth?
  $\rightarrow$ NO: cliques are EHF graphs
- Bounded rankwidth?
  $\rightarrow$ NO: a set of diamond-free EHF graphs (constructed by Adler, et.al.)

Theorem 1[a]

[a] Cameron, da Silva, Huang, Vŭsković, 2016

Every triangle-free EHF graph has treewidth at most 5.

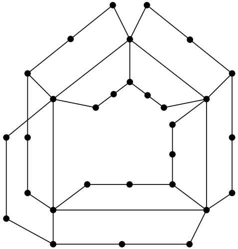

Figure: triangle-free EHF graph

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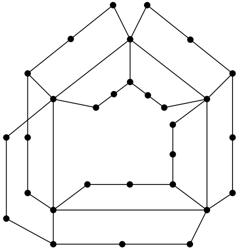

Figure: triangle-free EHF graph

Original motivation:

Does K_4-free EHF graphs have **bounded** treewidth?

We propose similar question for (theta, triangle)-free graphs.

Wheels with two non-adjacent centers:

- In triangle-free EHF graphs: always nested
- In (theta, triangle)-free graphs: nested, except the cube
- In K_4-free EHF graphs: nested with several exceptions

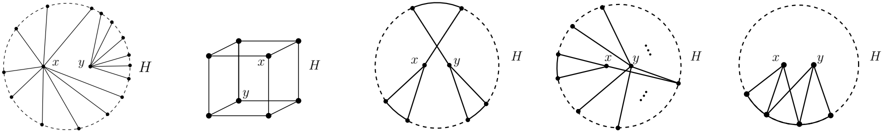

Figure: nested-wheel, cube, and several exceptions

Theorem 2

There exist:

1. (theta, triangle)-free graphs with arbitrarily large treewidth and rankwidth
2. $K4$-free EHF graphs with arbitrarily large treewidth and rankwidth

Remark. The graphs in Thm 2 are variants of ***layered wheels***

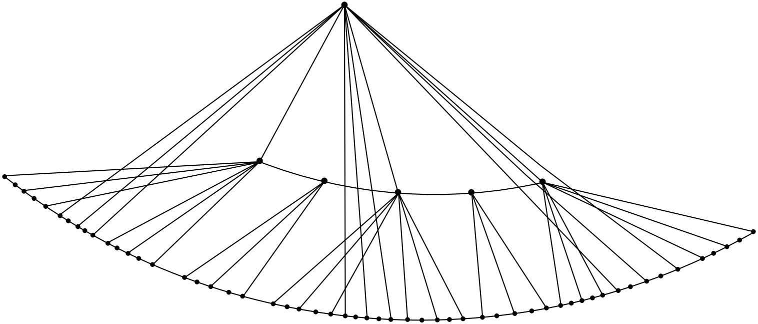

Figure: Layered wheel $G_{3,4}$

Layered wheel $G_{\ell,k}$ for $\ell \geq 1, k \geq 4$:

- it consists of ℓ **layers**
- it has **girth** equals to k

CONSTRUCTION:

$G(\ell, k)$, with $\ell = 3$ and $k = 4$

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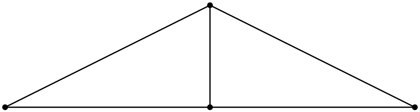

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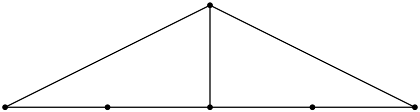

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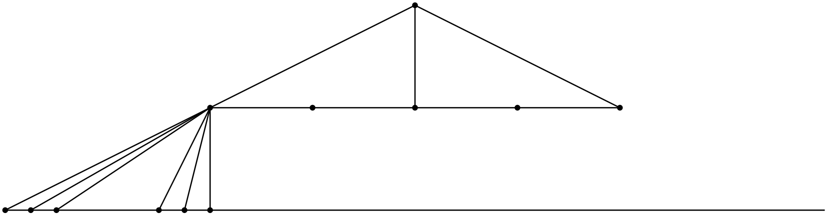

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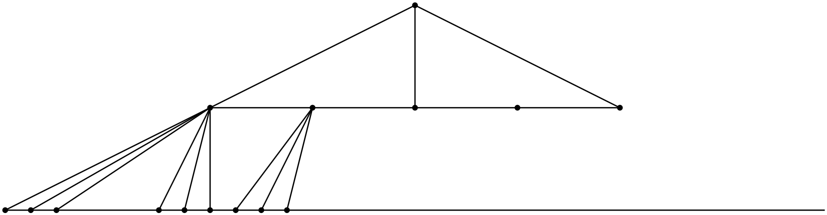

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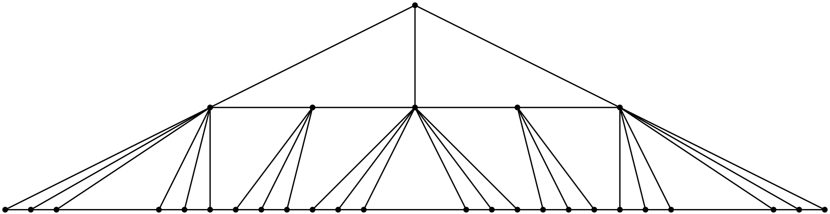

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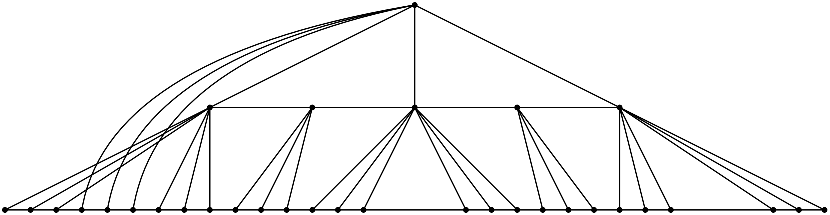

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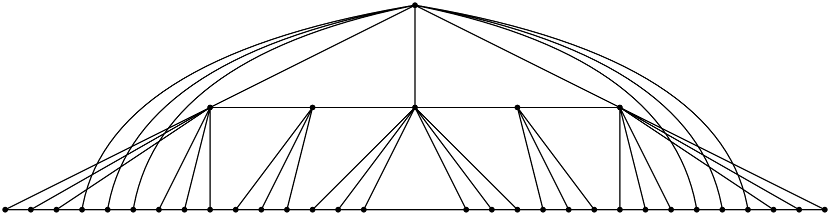

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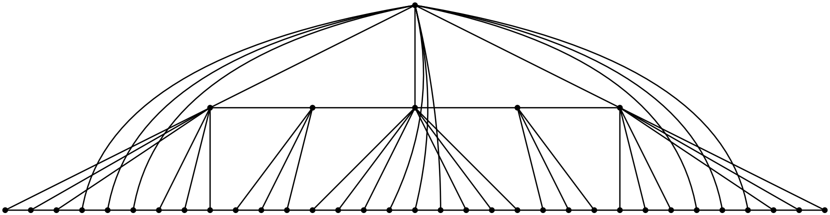

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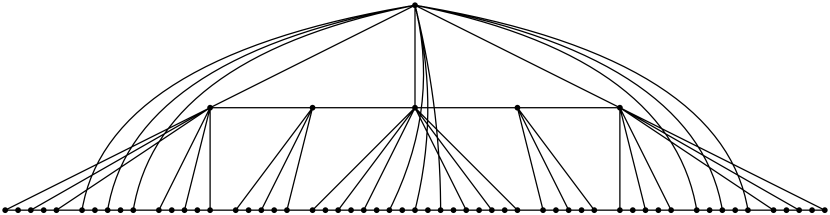


$G(\ell, k)$, with $\ell = 3$ and $k = 4$

Some properties of layered wheel:

- Every vertex has neighbors in the next layers
- Every vertex has at most one ancestor
- The last layer contains vertices of degree 2 (hence, it is 3-colorable)

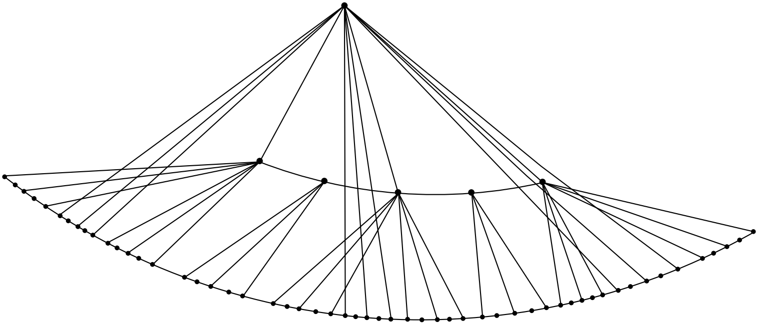


Figure: Layered wheel $G_{3,4}$

For $\ell \geq 1$, $k \geq 4$, layered wheel $G_{\ell,k}$ satisfies the following:

1. girth$(G_{\ell,k}) = k$

   by the rule of subdivision

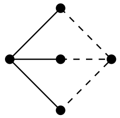
2. $tw(G_{\ell,k}) \geq \ell$

   because it contains a clique minor on ℓ vertices

3. $rw(G_{\ell,k}) \geq f(\ell)$, for some linear function ℓ

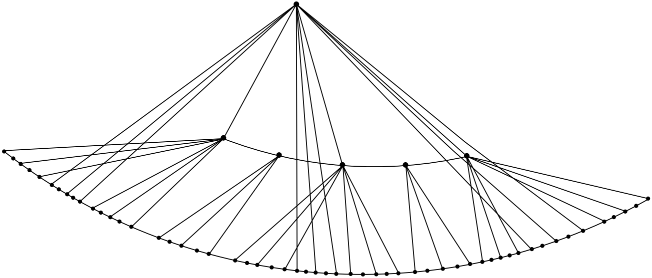
   our proof uses similar technique as for diamond-free EHF graphs

4. It does not contain a theta

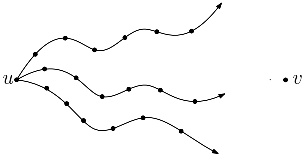


Proof sketch.

4. For $\ell \geq 1$, $k \geq 4$, $G_{\ell,k}$ does not contain a theta



$G_{\ell,k}$ is full of:



Lemma 1

For $\ell \geq 1$, $k \geq 4$, layered wheel $G_{\ell,k}$:

- contains r^ℓ vertices for some $r > 5$, and
- $\ell \leq tw(G_{\ell,k}) \leq 105\ell$.

So, $tw(G_{\ell,k}) \leq 105 \log_r |V(G_{\ell,k}|)$

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Proof sketch.

Lemma 2

Any subgraph of $G_{\ell,k}$ admits a $\frac{2}{3}$-*balanced separation* of order at most ℓ.

Theorem 3 [a]

[a] Dvořák & Norin, 2014

For any graph G, $tw(G) \leq 105 \cdot \ell$, where ℓ is the smallest number such that every subgraph of G admits a $\frac{2}{3}$-balanced separation of order $\leq \ell$.

Conjecture 1

For any (theta, triangle)-free graph G, and some constant $c > 0$,

$$tw(G) \leq c \cdot \log(|V(G)|).$$

Conjecture 2

Same conjecture for K_4-free EHF graphs.

Consequence.

A lot of graph optimization problems on (theta, triangle)-free graph are poly-time solvable.

In particular, given a tree decomposition of n-vertex graph G with width t, such a problem is solvable in time $t^{O(t)} \cdot n$.

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Proof plan.

Let $t > 0, r > 1$, and $\mathcal{F}_\ell$ be a set of graph such that:

1. for every $H \in \mathcal{F}_\ell$, we have $|V(H)| \geq r^\ell$
2. every (theta, triangle, $\mathcal{F}_\ell$)-free graph has treewidth at most $t \cdot \ell$.

Hence any (theta, triangle)-free graph G with $r^\ell \leq |V(G)| < r^{\ell+1}$ satisfies

$$tw(G) \leq t \cdot (\ell + 1) \leq t \cdot (\log_r |V(G)| + 1) \leq c \cdot \log |V(G)|$$

for some constant c.

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Remark. A possible candidate $\mathcal{F}_\ell = \{\text{layered wheels of } \ell \text{ layers}\}$.

$\rightarrow$ (1) is satisfied, how about (2)?

A weakening result: $\mathcal{F}_\ell = \{\ell\textbf{-wheels}\}$

An ℓ***-wheel*** is a graph formed by a hole H and a set X of ℓ vertices, such that (H, x) is a wheel for any $x \in X$.

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Theorem 4

For $\ell \geq 0$, any (theta, triangle, ℓ-wheel)-free graph has treewidth $O\Big(\big(\frac{c(\ell+2)^2}{\log(\ell+2)}\big)^{19}\mathrm{polylog}\big(\frac{c(\ell+2)^2}{\log(\ell+2)}\big)\Big)$ for some constant c.

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Problem:

- An ℓ-wheel contains only $\geq r \cdot \ell$ vertices for some constant r.
- It might be possible to improve the bound of Thm 4 into a linear function $f(\ell)$, but might not be better than that.

Proof sketch.

General tools to bound the treewidth:

- A *min-cut separation* of a graph H is a partition (A, C, B) of $V(H)$, where C is a cutset separating A and B, such that:
  - $H[A]$ and $H[B]$ are both non-empty and connected
  - Every vertex $v \in C$ has neighbor in both A and B

Lemma 3

If G is (theta, triangle, ℓ-wheel)-free, then any min-cut separation of $H \subseteq_{\mathrm{ind}} G$ has order $\leq \frac{c(\ell+2)^2}{\log(\ell+2)}$ for some constant c.

Lemma 4 [a]

[a] with Thomassé

Any graph G satisfying the following, has treewidth $\leq O((2\ell)^{19}\mathrm{polylog}(2\ell))$.

- it contains no clique $K_{2\ell}$, and
- every min-cut separation of $H \subseteq_{\mathrm{ind}} G$ has order $\leq \ell$.

Improvement

Lemma 4+ [a]

[a] Pilipczuk, April 2019

Any graph G satisfying the following, has treewidth $\leq (k-1)\ell^3 - 1$.

- it contains no clique K_k, and
- every min-cut separation of $H \subseteq_{\text{ind}} G$ has order $\leq \ell$.

Proof idea. Using a so-called *potential maximal clique* (PMC).

Theorem 4+

For $\ell \geq 0$, any (theta, triangle, ℓ-wheel)-free graph has treewidth $\leq 2 \cdot \left(\frac{c(\ell+2)^2}{\log(\ell+2)}\right)^3 - 1$ for some constant c.

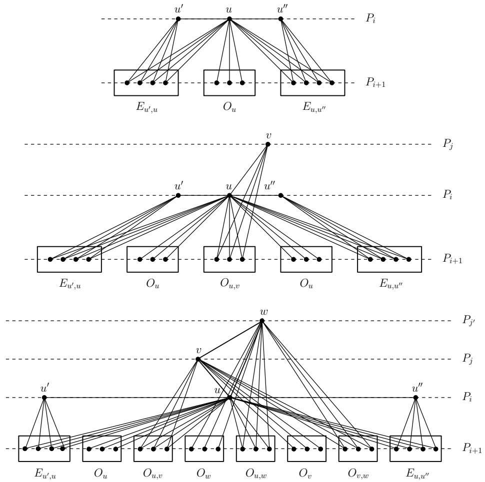

Figure: Construction of K_4-free EHF-layered-wheels

Theorem 4

For any $\ell \geq 1$, there exists an EHF-layered-wheel with treewidth $\geq \ell$ and rankwidth $\geq f(\ell)$ for some function f.

Remark. This answers the following question of Cameron et.al.:

> is the treewidth/cliquewidth of an EHF graphs
> bounded by a function of its clique number?

no, because EHF-layered-wheels are K_4-free and even-hole-free

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Question

What $\mathcal{F}_\ell$ could be?

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— THANK YOU FOR YOUR ATTENTION! —